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Worldsheet instanton corrections to the Kaluza-Klein monopole

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Abstract: The Kaluza-Klein monopole is a well known object in both gravity and string theory, related by T-duality to a "smeared" NS5-brane which retains the isometry around the duality circle. As the true NS5-brane solution is localized at a point on the circle, duality implies that the Kaluza-Klein monopole should show some corresponding behavior. In this paper, we express the Kaluza-Klein monopole as a gauged linear sigma model in two dimensions and show that worldsheet instantons give corrections to its geometry. These corrections can be understood as a localization in "winding space" which could be probed by strings with winding charge around the circle.

Keywords: Solitons Monopoles and Instantons, String Duality.

Contents

1. Introduction

It is well known that superstring theory compactified on a circle contains Kaluza-Klein monopoles [1, 2], which have an isometry around the circle. T-duality in that direction transforms them into H-monopoles [3, 4], which are understood in string theory as NS5-branes [5, 6]. This relationship forms an important part of the duality web.

However, there are two serious gaps in this familiar story. The first is that NS5-branes naturally correspond to the localized H-monopole geometry found in [5]. But that solution breaks the isometry around the circle and has a throat behavior at short distances (at least when the NS5-brane charge is greater than one). This is qualitatively very different from the Kaluza-Klein monopole solution, and thus would seem to conflict with the basic premise of T-duality that the physics of the dual solutions is the same.

On this basis, Gregory, Moore, and one of the authors [7] argued that the proper Kaluza-Klein monopole solution in string theory should be modified. They suggested that classical values for string winding states near the monopole core would lead to a "throat" that could be probed by scattering winding strings. However, they did not find the corrected geometry explicitly, and the mechanism generating the necessary corrections remained unknown.

The second gap in the story is that the natural T-dual of a Kaluza-Klein monopole geometry is a *smeared* H-monopole solution which retains the S^1 isometry. We know, however, that one of the H-monopole's moduli is its position on S^1. There must therefore be a localized solution, or else changing the S^1 location would not be a true physical modulus as it would not lead to a new point in the physical configuration space. This puzzle was resolved by Tong [8], who demonstrated that worldsheet instantons in the smeared H-monopole background correct the geometry to reproduce the localized solution of [5]. The solution to this second problem provides the necessary tools to solve the first.

In this paper, we show that the unit charge Kaluza-Klein monopole receives similar corrections from worldsheet instantons. As its topology does not admit holomorphic instantons, we begin from an $\mathcal{N} = (4,4)$ supersymmetric gauged linear sigma model and study the "point-like instantons" described by Witten [9]. Although we are only able to find the corrected solution in a limit, our results strongly suggest that the Kaluza-Klein monopole in string theory is localized in "winding space" as expected from T-duality. Our perspective on the corrections differs slightly from that of [7]: while they carry the same conserved charge as string winding states, there is no direct identification between the two.

The structure of this paper is as follows. In section 2 we briefly review the relevant monopole geometries. In section 3 we state the supersymmetric gauged linear sigma models describing the Kaluza-Klein and H-monopoles and show that the former reduces to the expected nonlinear sigma model in the low-energy limit. In section 4 we identify worldsheet instanton configurations in the gauged linear sigma model and determine their leading order effect on the geometry. Finally, in section 5 we relate these results to the expected properties of the solution and conclude.

2. Review of monopole geometries

The usual Kaluza-Klein monopole metric is that of Taub-NUT space, with no excitation of the antisymmetric tensor or dilaton. In our conventions,

$$ds^2 = H(r)\, d\boldsymbol{r} \cdot d\boldsymbol{r} + H(r)^{-1} \left(d\kappa + \frac{1}{2} \boldsymbol{\omega} \cdot d\boldsymbol{r} \right)^2 . \tag{2.1}$$

Here, $\boldsymbol{r}$ is a position in $\mathbb{R}^3$, $\boldsymbol{\omega}$ is a vector in $\mathbb{R}^3$ satisfying $\nabla \times \boldsymbol{\omega} = -2\nabla H(r) = -\nabla(1/r)$, and κ has period 2π. The harmonic function $H(r)$ is

$$H(r) = \frac{1}{g^2} + \frac{1}{2r} ,$$

making g the asymptotic radius of the κ circle (in string units).[1] This solution approaches flat space at the origin and has global topology $\mathbb{R}^4$: its local $\mathbb{R}^3 \times S^1$ structure is simply

[1] To compare with the more common conventions of [7], the radius is $g = R/\sqrt{\alpha'}$, the coordinates are $r_{\text{here}} = r_{\text{there}} \cdot R/\alpha'$ and $\kappa = x^5/R$, the metric is $ds^2_{\text{here}} = -ds^2_{\text{there}}/\alpha'$, and the harmonic function in the metric there is $U^{-1} = g^2 H = 1 + R/2r_{\text{there}}$.

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the Hopf fibration of S^1 over $S^2(\times\mathbb{R}_+)$. After dimensional reduction on the S^1, $\boldsymbol{\omega}$ gives the vector potential of a magnetic monopole for the Kaluza-Klein gauge field. In spherical coordinates $\{r,\vartheta,\varphi\}$ on $\mathbb{R}^3$, one common gauge choice gives $\omega_r=\omega_\vartheta=0$, $\omega_\varphi=1-\cos\vartheta$.

In addition to its three collective coordinates corresponding to position in $\mathbb{R}^3$, the Kaluza-Klein monopole has a fourth collective coordinate related to the antisymmetric 2-form B_{mn} [10]. This dyonic coordinate β arises from the harmonic 2-form of Taub-NUT space, and can be found as a large gauge transformation of B_{mn}:

$$\mathsf{B}=\beta\, d\Lambda\,, \qquad \text{for} \qquad \Lambda=\frac{1}{g^2H(r)}\left(d\kappa+\frac{1}{2}\boldsymbol{\omega}\cdot d\boldsymbol{r}\right). \tag{2.2}$$

Although it is pure gauge, this B is physically significant because Λ does not vanish at infinity. If β is constant, B is a closed form and has no effect on the geometry, but (for instance) a time-varying β carries string winding charge, which corresponds after dimensional reduction to electric charge under B_{m4} (where $r^4\equiv\kappa$).

The (smeared) H-monopole solution can be found from this by the usual Buscher rules for T-duality [11]. Applying them to the Kaluza-Klein monopole (with $\beta=0$) gives a solution in terms of the dual coordinate $r^4\equiv\theta$:

$$ds^2=H(r)\left(d\boldsymbol{r}\cdot d\boldsymbol{r}+d\theta^2\right)\,, \qquad B_{m4}=-\frac{\boldsymbol{\omega}}{2} \quad (m=1,2,3)\,,$$

with all other independent B_{mn} zero. (The dilaton becomes $e^\Phi=H(r)$.) The physical contribution of B_{mn} is the torsion $\mathsf{T}=-\mathsf{H}=-d\mathsf{B}$, which can be written in terms of $H(r)$: $H_{mnp}=\epsilon_{mnp}{}^qH^{-1}\partial_qH$. Applying the Buscher rules to the general solution with $\beta\neq0$ reduces to the same form after a coordinate transformation $\theta\to\theta-\beta/(g^2H(r))$ and a gauge transformation of B_{mn} (with a gauge parameter that does vanish at infinity). Thus, up to a coordinate transformation that is trivial at infinity, β simply corresponds to a shift in θ.

Finally, the localized H-monopole can be constructed from a periodic array of NS5-branes [5]. The forms of the metric and torsion are the same as in the smeared case above, but the harmonic function is modified:

$$H(r,\theta)=\frac{1}{g^2}+\frac{1}{2r}\frac{\sinh r}{\cosh r-\cos\theta}=\frac{1}{g^2}+\frac{1}{2r}\sum_{k=-\infty}^{\infty}e^{-|k|r+ik\theta}\,. \tag{2.3}$$

The Fourier-expanded form is directly related to the instanton sum. In these coordinates the monopole is localized at $\theta=0$; more generally, we can introduce a constant offset $(\theta-\theta_0)$.

3. Sigma model actions for monopoles

To study these monopole configurations in string theory, our first step must be to identify the appropriate worldsheet theories describing them. These geometries contain no holomorphic curves (at least for unit monopole charge), so we will consider worldsheet instanton effects due to point-like instantons in gauged linear sigma models [9]. The appropriate models have $\mathcal{N}=(4,4)$ supersymmetry in two dimensions.

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Our approach in this section generally follows that of Tong [8], although our conventions and methods differ slightly. We introduce the T-dual monopole actions in $\mathcal{N} = (2, 2)$ superspace after defining the necessary superfields. We then expand the actions in components as will be necessary for the instanton calculation in section 4. To make contact with the Kaluza-Klein geometry given above, we then take the low energy limit and show that it reduces to the expected nonlinear sigma model. This limit will be used in interpreting the results of the instanton calculation.

3.1 Superfield definitions

The gauged linear sigma model actions for the H-monopole and the Kaluza-Klein monopole are constructed from $\mathcal{N} = (4, 4)$ supermultiplets in two dimensions. Each action includes a gauge multiplet and a charged hypermultiplet, but where the H-monopole also has a twisted hypermultiplet the Kaluza-Klein monopole has a normal hypermultiplet instead. For ease of computation, we decompose each of these $\mathcal{N} = (4, 4)$ supermultiplets into a pair of $\mathcal{N} = (2, 2)$ superfields. (See [9] for a review of these models.)

We begin with the $\mathcal{N} = (4, 4)$ vector multiplet, which decomposes into an $\mathcal{N} = (2, 2)$ chiral superfield Φ and vector superfield V. In terms of component fields (with derivative terms suppressed), these are

$$\begin{aligned}
\Phi &= \phi + \sqrt{2}\,\theta^+\tilde{\lambda}_+ + \sqrt{2}\,\theta^-\tilde{\lambda}_- + \sqrt{2}\,\theta^+\theta^-(D^1 - iD^2) + \cdots \\
V &= \theta^+\bar{\theta}^+A_+ + \theta^-\bar{\theta}^-A_- - \sqrt{2}\,\theta^-\bar{\theta}^+\sigma - \sqrt{2}\,\theta^+\bar{\theta}^-\sigma^\dagger - \\
&\quad - 2i\theta^+\theta^-\left(\bar{\theta}^-\bar{\lambda}_- + \bar{\theta}^+\bar{\lambda}_+\right) - 2i\bar{\theta}^-\bar{\theta}^+\left(\theta^+\lambda_+ + \theta^-\lambda_-\right) + 2\theta^+\theta^-\bar{\theta}^-\bar{\theta}^+D^3 \\
\Sigma &\equiv \frac{1}{\sqrt{2}}\bar{D}_+D_-V = \sigma + i\sqrt{2}\,\theta^+\bar{\lambda}_+ - i\sqrt{2}\,\bar{\theta}^-\lambda_- + \sqrt{2}\,\theta^+\bar{\theta}^-(D^3 - iF_{01}) + \cdots\,.
\end{aligned}$$

Here, the vector superfield is in Wess-Zumino gauge and we have defined $A_\pm \equiv A_0 \pm A_1$. The gauge invariant twisted chiral superfield Σ allows a natural expression for the theta angle and Fayet-Iliopoulos term and provides a convenient way of writing the gauge kinetic term. Following [9], components of fermion doublets are labeled as $\lambda^\alpha = (\lambda^-, \lambda^+)$ and $\lambda_\alpha = (\lambda_-, \lambda_+)$, so $\lambda^- = \lambda_+$ and $\lambda^+ = -\lambda_-$. As usual, gauge transformations are given by $V \to V + i(\Lambda - \Lambda^\dagger)$ for an arbitrary chiral superfield Λ.

The charged hypermultiplet common to both monopole actions decomposes into two chiral superfields Q and $\tilde{Q}$ with charges $+1$ and -1 under the U(1) gauge group, respectively. Their component expansions are

$$\begin{aligned}
Q &= q + \sqrt{2}\,\theta^+\psi_+ + \sqrt{2}\,\theta^-\psi_- + 2\theta^+\theta^-F + \cdots \\
\tilde{Q} &= \tilde{q} + \sqrt{2}\,\theta^+\tilde{\psi}_+ + \sqrt{2}\,\theta^-\tilde{\psi}_- + 2\theta^+\theta^-\tilde{F} + \cdots\,.
\end{aligned}$$

The H-monopole action's twisted hypermultiplet can be decomposed into a chiral superfield Ψ and a twisted chiral superfield Θ. Their component expansions are

$$\begin{aligned}
\Psi &= \frac{(r^1 + ir^2)}{\sqrt{2}} + \sqrt{2}\,\theta^+\chi_+ + \sqrt{2}\,\theta^-\chi_- + 2\theta^+\theta^-G + \cdots \\
\Theta &= \frac{(r^3 + i\theta)}{\sqrt{2}} - i\sqrt{2}\,\theta^+\bar{\tilde{\chi}}_+ - i\sqrt{2}\,\bar{\theta}^-\tilde{\chi}_- + 2\theta^+\bar{\theta}^-\tilde{G} + \cdots\,.
\end{aligned}$$

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Under T-duality, this twisted hypermultiplet is exchanged for the normal hypermultiplet of the Kaluza-Klein monopole. The Ψ superfield is unchanged, but its partner is now a second chiral superfield Γ whose component expansion is

$$g^2\Gamma = \frac{(-r^3 + ig^2\gamma)}{\sqrt{2}} + i\sqrt{2}\,\theta^+\bar{\tilde{\chi}}_+ + i\sqrt{2}\,\theta^-\bar{\tilde{\chi}}_- + 2\theta^+\theta^-\tilde{G}' + \cdots . \tag{3.1}$$

Some of the component fields listed here share their names with components of Θ; these identifications are justified below.

3.2 Gauged linear sigma models and T-duality

3.2.1 The monopole actions in superspace

The gauged linear sigma model action corresponding to the H-monopole written in terms of the above $\mathcal{N} = (2,2)$ superfields is given by the sum of the following lagrangian densities (plus complex conjugates of the F and $\tilde{F}$ terms):

$$\begin{aligned}
\mathcal{L}_D &= \int d^4\theta \left[\frac{1}{e^2}\left(-\Sigma^\dagger\Sigma + \Phi^\dagger\Phi\right) + \frac{1}{g^2}\left(-\Theta^\dagger\Theta + \Psi^\dagger\Psi\right) + Q^\dagger e^{2V} Q + \tilde{Q}^\dagger e^{-2V}\tilde{Q}\right] \\
\mathcal{L}_F &= \int d^2\theta \left(\sqrt{2}\,\tilde{Q}\Phi Q - \Phi\Psi\right) \qquad \mathcal{L}_{\tilde{F}} = -\int d^2\vartheta\ \Theta\Sigma\,.
\end{aligned}$$

Here, $d^2\theta \equiv -d\theta^+ d\theta^-/2$ and $d^2\vartheta \equiv -d\theta^+ d\bar{\theta}^-/2$ are the usual measures on chiral and twisted chiral superspace, respectively. We begin from the H-monopole action in order to follow the effects of $\mathcal{L}_{\tilde{F}}$ through T-duality. As shown below, this term leads to a topologically significant total derivative of component fields after duality.

To obtain the corresponding superspace action for the Kaluza-Klein monopole, we find the T-dual action in superspace following Roček and Verlinde [12]. It is first necessary to write the action uniformly as an integral over full superspace by applying the identity

$$\int d^2\vartheta\ \Sigma\Theta + \int d^2\bar{\vartheta}\ \Sigma^\dagger\Theta^\dagger = \int d^4\theta \left[\sqrt{2}\left(\Theta + \Theta^\dagger\right)V\right] - \epsilon^{\mu\nu}\partial_\mu(\theta A_\nu)\,. \tag{3.2}$$

We can then write the Θ-dependent part of the action in first order form, replacing $\Theta + \Theta^\dagger$ with a real superfield B together with a chiral superfield Lagrange multiplier Γ:

$$\begin{aligned}
\int d^4\theta \left[-\frac{1}{g^2}\Theta^\dagger\Theta - \sqrt{2}\left(\Theta + \Theta^\dagger\right)V\right] &= \int d^4\theta \left[-\frac{1}{2g^2}\left(\Theta + \Theta^\dagger\right)^2 - \sqrt{2}\left(\Theta + \Theta^\dagger\right)V\right] \\
&= \int d^4\theta \left[-\frac{1}{2g^2}B^2 - \sqrt{2}\,BV - \left(\Gamma + \Gamma^\dagger\right)B\right].
\end{aligned}$$

The final equality can be proved by integrating out Γ (and its conjugate), which requires that we rewrite the Γ term as an integral over chiral superspace only. Up to total derivatives, $2\int d^2\bar{\theta}$ is equivalent to $-\bar{D}_+\bar{D}_-$. Both of these derivatives annihilate Γ by definition, so the Γ equation of motion is $\bar{D}_+\bar{D}_-B = 0$. The $\Gamma^\dagger$ constraint is just the conjugate of this, so we can write $B = \Theta + \Theta^\dagger$ for a twisted chiral superfield Θ (for which, by definition,

$\bar{D}_+\Theta = D_-\Theta = 0$). This combination $\Theta+\Theta^\dagger$ has no undifferentiated imaginary scalar part, so the only source for a constant offset in θ in the first order action is the total derivative term from eq. (3.2).

To find the T-dual action we instead integrate out B, which yields the equation of motion $B = -g^2(\Gamma + \Gamma^\dagger + \sqrt{2}\,V)$. This leaves us with the duality substitution

$$\int d^4\theta \left[-\frac{1}{g^2}\Theta^\dagger\Theta - \sqrt{2}\left(\Theta + \Theta^\dagger\right)V\right] \to \int d^4\theta\, \frac{g^2}{2}\left(\Gamma + \Gamma^\dagger + \sqrt{2}\,V\right)^2 .$$

The full superspace action for the Kaluza-Klein monopole is then constructed from

$$\begin{aligned}
\mathcal{L}_D &= \int d^4\theta \left[\frac{1}{e^2}\left(-\Sigma^\dagger\Sigma + \Phi^\dagger\Phi\right) + \frac{g^2}{2}\left(\Gamma+\Gamma^\dagger+\sqrt{2}\,V\right)^2 + \frac{1}{g^2}\Psi^\dagger\Psi + Q^\dagger e^{2V}Q + \tilde{Q}^\dagger e^{-2V}\tilde{Q}\right] \\
\mathcal{L}_F &= \int d^2\theta\, \left(\sqrt{2}\,\tilde{Q}\Phi Q - \Phi\Psi\right) \qquad\qquad \mathcal{L}_{\text{top.}} = \epsilon^{\mu\nu}\partial_\mu(\theta A_\nu)\,. \qquad (3.3)
\end{aligned}$$

In order for the action to remain gauge invariant, Γ must transform by a simple shift: $\Gamma \to \Gamma - i\sqrt{2}\,\Lambda$. In the case of an "ordinary" gauge transformation $\Lambda = \lambda \in \mathbb{R}$ (the residual freedom after fixing Wess-Zumino gauge), the shift affects only one component: $\gamma \to \gamma - 2\lambda$. The total derivative term $\mathcal{L}_{\text{top.}}$ will be topologically significant in the instanton calculation.

The component fields of Θ and Γ can be related to one another by equating the two expressions for B found above: $\Theta+\Theta^\dagger = B = -g^2(\Gamma+\Gamma^\dagger+\sqrt{2}\,V)$. This justifies the equality between the Θ and Γ component fields r^3 and $\tilde{\chi}_\pm$ asserted in eq. (3.1). The components θ and γ are not directly related, but their derivatives are: $\partial_\pm\theta/g = \mp g(\partial_\pm\gamma + A_\pm)$. The relative $\mp$ is the usual sign change of the right-moving worldsheet coordinate under T-duality, and the factors of g convert angles to arc lengths.

3.2.2 The monopole actions in components

When these superspace actions are expanded in components and the auxiliary fields are eliminated, the results are almost identical. Below, we present the results for the Kaluza-Klein monopole; the few changes required for the H-monopole are discussed in the text.

We divide the action into a sum of kinetic, scalar potential, "Yukawa," and topological terms:

$$S = \frac{1}{2\pi}\int d^2x\, \left(\mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{Yuk}} + \mathcal{L}_{\text{top.}}\right).$$

The term $\mathcal{L}_{\text{top.}}$ was defined in eq. (3.3); it is absent from the H-monopole action. The component form of the kinetic terms is:

$$\begin{aligned}
\mathcal{L}_{\text{kin}} = &\frac{1}{e^2}\left(\frac{1}{2}F_{01}^2 - |\partial_\mu\phi|^2 - |\partial_\mu\sigma|^2 + i(\bar{\lambda}_+\partial_-\lambda_+ + \bar{\tilde{\lambda}}_+\partial_-\tilde{\lambda}_+ + \bar{\lambda}_-\partial_+\lambda_- + \bar{\tilde{\lambda}}_-\partial_+\tilde{\lambda}_-)\right) + \\
&+ \frac{1}{g^2}\left(-\frac{1}{2}|\partial_\mu\boldsymbol{r}|^2 - \frac{g^4}{2}(\partial_\mu\gamma + A_\mu)^2 + i(\bar{\chi}_+\partial_-\chi_+ + \bar{\tilde{\chi}}_+\partial_-\tilde{\chi}_+ + \bar{\chi}_-\partial_+\chi_- + \bar{\tilde{\chi}}_-\partial_+\tilde{\chi}_-)\right) + \\
&+ \left(-|\mathcal{D}_\mu q|^2 - |\mathcal{D}_\mu\tilde{q}|^2 + i(\bar{\psi}_+\mathcal{D}_-\psi_+ + \bar{\tilde{\psi}}_+\mathcal{D}_-\tilde{\psi}_+ + \bar{\psi}_-\mathcal{D}_+\psi_- + \bar{\tilde{\psi}}_-\mathcal{D}_+\tilde{\psi}_-)\right). \qquad (3.4)
\end{aligned}$$

The worldsheet metric is flat and given by $\eta_{\mu\nu} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have defined $\partial_\pm = \partial_0 \pm \partial_1$, and the covariant derivative is defined as $\mathcal{D}_\mu q = \partial_\mu q + iA_\mu q$ and $\mathcal{D}_\mu \tilde{q} = \partial_\mu \tilde{q} - iA_\mu \tilde{q}$ (and similarly for other components from the same supermultiplet). To obtain the kinetic terms of the H-monopole action, only one substitution is required: $-\frac{g^4}{2}(\partial_\mu\gamma + A_\mu)^2 \to -\frac{1}{2}(\partial_\mu\theta)^2$.

After eliminating auxiliary fields, the scalar potential is:

$$
\begin{aligned}
\mathcal{L}_{\text{pot}} = & -\frac{e^2}{2}\left(|q|^2 - |\tilde{q}|^2 - r^3\right)^2 - \frac{e^2}{2}\left|2\tilde{q}q - \left(r^1 + ir^2\right)\right|^2 - \\
& - \left(|\phi|^2 + |\sigma|^2\right)\left(g^2 + 2|q|^2 + 2|\tilde{q}|^2\right).
\end{aligned} \tag{3.5}
$$

The H-monopole action also includes the term $-\theta F_{01}$.

Finally, the "Yukawa" action (which also includes several two-fermion terms) is:

$$
\begin{aligned}
\mathcal{L}_{\text{Yuk}} = & \left(\tilde{\lambda}_+\chi_- - \lambda_+\bar{\tilde{\chi}}_- + \bar{\lambda}_+\tilde{\chi}_- - \bar{\tilde{\lambda}}_+\bar{\chi}_-\right) - \left(\tilde{\lambda}_-\chi_+ - \lambda_-\bar{\tilde{\chi}}_+ + \bar{\lambda}_-\tilde{\chi}_+ - \bar{\tilde{\lambda}}_-\bar{\chi}_+\right) + \\
& + i\sqrt{2}\, q\left(\bar{\lambda}_+\bar{\psi}_- - \bar{\lambda}_-\bar{\psi}_+ + i\tilde{\lambda}_+\tilde{\psi}_- - i\tilde{\lambda}_-\tilde{\psi}_+\right) + \sqrt{2}\,\sigma\,\left(\psi_+\bar{\psi}_- - \tilde{\psi}_+\bar{\tilde{\psi}}_-\right) + \\
& + i\sqrt{2}\, q^\dagger\left(\lambda_+\psi_- - \lambda_-\psi_+ - i\bar{\tilde{\lambda}}_+\bar{\tilde{\psi}}_- + i\bar{\tilde{\lambda}}_-\bar{\tilde{\psi}}_+\right) - \sqrt{2}\,\sigma^\dagger\left(\bar{\psi}_+\psi_- - \bar{\tilde{\psi}}_+\tilde{\psi}_-\right) - \\
& - i\sqrt{2}\,\tilde{q}\left(\bar{\lambda}_+\bar{\tilde{\psi}}_- - \bar{\lambda}_-\bar{\tilde{\psi}}_+ - i\tilde{\lambda}_+\psi_- + i\tilde{\lambda}_-\psi_+\right) - \sqrt{2}\,\phi\,\left(\psi_+\tilde{\psi}_- + \tilde{\psi}_+\psi_-\right) - \\
& - i\sqrt{2}\,\tilde{q}^\dagger\left(\lambda_+\tilde{\psi}_- - \lambda_-\tilde{\psi}_+ + i\bar{\tilde{\lambda}}_+\bar{\psi}_- - i\bar{\tilde{\lambda}}_-\bar{\psi}_+\right) + \sqrt{2}\,\phi^\dagger\left(\bar{\psi}_+\bar{\tilde{\psi}}_- + \bar{\tilde{\psi}}_+\bar{\psi}_-\right).
\end{aligned}
$$

The corresponding terms in the H-monopole action are identical.

Both actions are invariant under the R-symmetry group $\mathrm{SU}(2)\times\mathrm{SO}(4) = [\mathrm{SU}(2)]^3$. The component fields fall into R-multiplets, which we label by their structure under $\mathrm{SU}(2)_V \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. We define the scalar R-multiplets as $q_i = (q, \tilde{q}^\dagger)$ (that is, a $(\mathbf{2},\mathbf{1},\mathbf{1})$) and $r^m = (r^1, r^2, r^3)$ (a $(\mathbf{3},\mathbf{1},\mathbf{1})$). θ and γ are R-singlets, and the vector multiplet scalars fall into a $(\mathbf{1},\mathbf{2},\mathbf{2})$ that obeys a reality condition. The fermions also fall into R-multiplets: a $(\mathbf{1},\mathbf{2},\mathbf{1})$ and a $(\mathbf{1},\mathbf{1},\mathbf{2})$ for the ψs, real multiplets $(\mathbf{2},\bar{\mathbf{2}},\mathbf{1})$ and $(\mathbf{2},\mathbf{1},\bar{\mathbf{2}})$ for the χs, and real multiplets $(\mathbf{2},\mathbf{2},\mathbf{1})$ and $(\mathbf{2},\mathbf{1},\mathbf{2})$ for the λs.

Finally, a quadratic combination of the χs into a $(\mathbf{3},\mathbf{1},\mathbf{1})$ multiplet arises in several places:

$$
(\chi\chi)^m_\pm \equiv \left(i\left(\chi_\pm\tilde{\chi}_\pm + \bar{\chi}_\pm\bar{\tilde{\chi}}_\pm\right),\quad \left(\chi_\pm\tilde{\chi}_\pm - \bar{\chi}_\pm\bar{\tilde{\chi}}_\pm\right),\quad \left(\bar{\chi}_\pm\chi_\pm + \bar{\tilde{\chi}}_\pm\tilde{\chi}_\pm\right)\right). \tag{3.6}
$$

3.3 Low energy limit of the Kaluza-Klein monopole action

3.3.1 The low energy limit in superspace

Nonlinear sigma models for the H-monopole and Kaluza-Klein monopole can be found as the low energy limits of these gauged linear sigma models, which can be taken in two ways. The first is based on the component action as given above. In our conventions, dimensional analysis shows that the only dimensionful parameter in this action is the gauge coupling e. The low energy limit is thus $e^2 \to \infty$, so the gauge kinetic terms vanish and both of the q–r terms in the scalar potential must vanish to ensure finite energy. The vector multiplet components become auxiliary fields and must be integrated out, leaving an action which can be written in terms of twisted hypermultiplet fields alone (after applying the constraints to eliminate the hypermultiplet fields). This is essentially the approach taken by Tong.

The second approach, which we will follow, is to take the $e^2 \to \infty$ limit while still in the superspace formalism. The kinetic terms for V (written in terms of Σ) and Φ vanish in eq. (3.3), so both can be treated as auxiliary superfields and integrated out directly. The superfield Φ appears only as a Lagrange multiplier in $\mathcal{L}_F$. The vector superfield is somewhat more subtle, as we must deal with the issue of gauge fixing before integrating it out. We first restrict to Wess-Zumino gauge, which fixes all of the gauge freedom except the ordinary gauge transformations of the component A_μ. The gauge choice for this residual symmetry is less crucial; when it is necessary to make an explicit choice, we will require that q be purely negative imaginary ($q = -ip$ for real $p > 0$).

The vector multiplet equations of motion resulting from this procedure are

$$\Psi = \sqrt{2}\,\tilde{Q}Q \qquad \text{and} \qquad \frac{g^2}{\sqrt{2}}\left(\Gamma + \Gamma^\dagger\right) = -Q^\dagger e^{2V} Q + \tilde{Q}^\dagger e^{-2V}\tilde{Q} - g^2 V\,. \tag{3.7}$$

It can be verified that these constraints contain precisely the same information as the vacuum equations and auxiliary field equations of motion in the component formalism.

When we do go to components, we eventually want to express the full action in terms of the twisted hypermultiplet fields, but as an intermediate step, we can (partially) apply the constraints directly to the Kaluza-Klein monopole action in superspace:

$$\begin{aligned}
\mathcal{L} &= \int d^4\theta \left[\frac{g^2}{2}\left(\Gamma + \Gamma^\dagger + \sqrt{2}\,V\right)^2 + \frac{1}{g^2}\Psi^\dagger\Psi + Q^\dagger e^{2V}Q + \tilde{Q}^\dagger e^{-2V}\tilde{Q}\right] \\
&= \int d^4\theta \left[g^2\Gamma^\dagger\Gamma + \sqrt{2}\,g^2 V\left(\Gamma + \Gamma^\dagger\right) + g^2V^2 + \frac{1}{g^2}\Psi^\dagger\Psi + Q^\dagger e^{2V}Q + \tilde{Q}^\dagger e^{-2V}\tilde{Q}\right] \\
&= \int d^4\theta \left[g^2\Gamma^\dagger\Gamma - g^2V^2 + \frac{1}{g^2}\Psi^\dagger\Psi + \left(1 - 2V^2\right)\left(Q^\dagger Q + \tilde{Q}^\dagger\tilde{Q}\right)\right].
\end{aligned} \tag{3.8}$$

In the last line, we have used our choice of Wess-Zumino gauge to expand the exponentials. The total derivative term $\mathcal{L}_{\text{top.}}$ is still present as well, but is not relevant to these manipulations in superspace. Completely eliminating the Q and V superfields at this point would be very difficult due to the form of the constraints, so the final constraint substitutions will be carried out at the component level.

3.3.2 The low energy limit in components

The nonlinear sigma model for the Kaluza-Klein monopole is found from this result by expanding eq. (3.8) in components and then applying the constraints from eq. (3.7). The constraints used to eliminate the Q and $\tilde{Q}$ components in favor of those from Ψ and Γ are

$$\begin{aligned}
r^1 + ir^2 &= 2q\tilde{q} \qquad & \chi_\pm &= \sqrt{2}\left(\tilde{q}\psi_\pm + q\tilde{\psi}_\pm\right) \\
r^3 &= |q|^2 - |\tilde{q}|^2 \qquad & \tilde{\chi}_\pm &= i\sqrt{2}\left(\tilde{q}\bar{\tilde{\psi}}_\pm - q\bar{\psi}_\pm\right).
\end{aligned} \tag{3.9}$$

These same relationships are found in the H-monopole case. It is also useful to express q and $\tilde{q}$ in terms of the r^m, but doing so is complicated by the fact that together, q and $\tilde{q}$ have four real degrees of freedom while r^m has only three. We choose to treat the phase of q as the fourth degree of freedom, so

$$q = -\frac{i}{\sqrt{2}} e^{-i\alpha} \sqrt{r + r^3}\,, \qquad \tilde{q} = \frac{i}{\sqrt{2}} e^{i\alpha} \frac{r^1 + ir^2}{\sqrt{r + r^3}}\,. \tag{3.10}$$

This choice of α is convenient when we impose our gauge condition: $q = -ip$ ($p > 0$) corresponds to $\alpha = 0$. The effect of an ordinary gauge transformation $q_i \to e^{-2i\lambda} q_i$ is $\alpha \to \alpha + 2\lambda$; $\boldsymbol{r}$ remains gauge invariant. This leads to a natural gauge invariant combination of real scalars

$$\kappa \equiv \gamma + \alpha\,.$$

This new scalar κ is a good choice for the coordinate r^4 of the Kaluza-Klein monopole: gauge variant coordinate fields have more complicated supersymmetry transformations.

The constraints also contain information on the auxiliary fields. The constraint involving $A_\pm$ is the most notable of these:

$$\begin{aligned} A_\pm &= \frac{1}{g^2 + 2r} \left(-g^2 \partial_\pm \gamma + r\boldsymbol{\omega} \cdot \partial_\pm \boldsymbol{r} + 2r\partial_\pm \alpha + 2(\bar{\psi}_\pm \psi_\pm - \bar{\tilde{\psi}}_\pm \tilde{\psi}_\pm) \right) \\ &= \frac{1}{g^2 H} \left(\partial_\pm \kappa + \frac{1}{2} \boldsymbol{\omega} \cdot \partial_\pm \boldsymbol{r} + \frac{1}{r} (\bar{\psi}_\pm \psi_\pm - \bar{\tilde{\psi}}_\pm \tilde{\psi}_\pm) \right) - \partial_\pm \gamma\,. \end{aligned} \tag{3.11}$$

This equation uses the (smeared) harmonic function $H = H(r)$ introduced in section 2 and the target space vector $\boldsymbol{\omega}$ defined (implicitly) by:

$$i \left(q^\dagger \partial_\mu q - q \partial_\mu q^\dagger - \tilde{q}^\dagger \partial_\mu \tilde{q} + \tilde{q} \partial_\mu \tilde{q}^\dagger \right) = \frac{r^1 \partial_\mu r^2 - r^2 \partial_\mu r^1}{r + r^3} + 2r\partial_\mu \alpha \equiv r\boldsymbol{\omega} \cdot \partial_\mu \boldsymbol{r} + 2r\partial_\mu \alpha\,. \tag{3.12}$$

In the first equality we have used eq. (3.10), but the final expression holds for redefinitions of α as described below. The explicit form of $\boldsymbol{\omega}$ shown here is simply the cartesian form of the monopole gauge field mentioned in section 2: $\omega_r = \omega_\vartheta = 0$, $\omega_\varphi = 1 - \cos\vartheta$. While $\boldsymbol{\omega}$ does not naturally change under gauge transformations, α does: $\delta(2r\partial_\mu \alpha) = 4r\nabla\lambda \cdot \partial_\mu \boldsymbol{r}$. If we redefine α and $\boldsymbol{\omega}$ so that $\alpha = 0$ again holds, $\boldsymbol{\omega}$ changes by $\delta\omega_\mu = 4\nabla_\mu \lambda$, a target space gauge transformation.

We can at last write the low energy action in components. After applying the constraints, the action finally reads as follows:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} H |\partial_\mu \boldsymbol{r}|^2 - \frac{1}{2} H^{-1} \left(\partial_\mu \kappa + \frac{1}{2} \boldsymbol{\omega} \cdot \partial_\mu \boldsymbol{r} \right)^2 + \\ & + iH \left(\bar{\chi}_+ \partial_- \chi_+ + \bar{\tilde{\chi}}_+ \partial_- \tilde{\chi}_+ + \bar{\chi}_- \partial_+ \chi_- + \bar{\tilde{\chi}}_- \partial_+ \tilde{\chi}_- \right) - \\ & - \frac{1}{4H|r|^3} \left(\partial_- \kappa + \frac{1}{2} \boldsymbol{\omega} \cdot \partial_- \boldsymbol{r} \right) r^m (\chi\chi)^m_+ - \frac{1}{4H|r|^3} \left(\partial_+ \kappa + \frac{1}{2} \boldsymbol{\omega} \cdot \partial_+ \boldsymbol{r} \right) r^m (\chi\chi)^m_- + \\ & + \frac{1}{4|r|^3} \epsilon_{mnp} (\chi\chi)^m_+ \, r^n \partial_- r^p + \frac{1}{4|r|^3} \epsilon_{mnp} (\chi\chi)^m_- \, r^n \partial_+ r^p - \\ & - \frac{3}{4g^2 H |r|^5} r^m (\chi\chi)^m_+ \, r^n (\chi\chi)^n_- + \frac{1}{4g^2 H |r|^3} (\chi\chi)^m_+ \, (\chi\chi)^m_-\,. \end{aligned} \tag{3.13}$$

Here, H is the harmonic function $H(r)$ defined in section 2, and the scalar fields γ and α have everywhere combined into κ. The vector combination $(\chi\chi)^m_\pm$ was defined in eq. (3.6).

In deriving this action, we have dropped total derivative terms, but the term $\mathcal{L}_{\text{top.}}$ remains significant:

$$\mathcal{L}_{\text{top.}} = \epsilon^{\mu\nu}\partial_\mu(\theta A_\nu) = \theta\epsilon^{\mu\nu}\partial_\mu A_\nu + \epsilon^{\mu\nu}\partial_\mu\theta A_\nu\,, \tag{3.14}$$

where A_ν should be replaced by its equation of motion in eq. (3.11). The bosonic part of $\partial_\mu\gamma + A_\mu$ is precisely the large gauge transformation 1-form Λ introduced in section 2, so the first of these terms has the form of the dyonic B-field of Taub-NUT: $\mathsf{B} = -\theta d\Lambda$. (The identification of this term in the action as a B-field follows from eq. (3.19) below.) Thus, although θ loses its geometrical meaning after T-duality, it is still significant as the dyonic coordinate of the Kaluza-Klein monopole.

The second term is harder to interpret, as the action does not treat θ as a dynamical field; if θ is constant, this term vanishes. It would be interesting to find an action which correctly encoded independent dynamics for a geometrical coordinate (κ, here) and its dual coordinate (θ), but no such formalism is currently known.

3.3.3 Real superfields and the nonlinear sigma model action

To find the geometric meaning of these results, we can rewrite the action as a supersymmetric nonlinear sigma model in terms of real $\mathcal{N} = (1,1)$ superfields whose scalar parts are the coordinate fields. We use conventions in which the $\mathcal{N} = (1,1)$ supercoordinates are pure imaginary, $\bar\theta^\alpha = -\theta^\alpha$, so the component expansion of a real superfield $R = R^\dagger$ is

$$R = A + \sqrt{2}\,\theta^+\Omega_+ + \sqrt{2}\,\theta^-\Omega_- + i\theta^+\theta^- F\,.$$

Here, A and F are real scalars and Ω is a real spinor.

To extract real superfields from our chiral superfields Ψ and Γ, we simply take their real and imaginary parts and impose the pure imaginary condition on θ^α (essentially setting the real part of our $\mathcal{N} = (2,2)$ θ^αs to zero). So, for instance, $\Psi_1 \equiv (\Psi + \Psi^\dagger)/\sqrt{2} = r^1 + \cdots$ and $-g^2\Gamma_1 \equiv -g^2(\Gamma + \Gamma^\dagger)/\sqrt{2} = r^3 + \cdots$. From these coordinate superfields, we can read off the real fermions Ω^m naturally associated to each of the coordinates r^m and to γ:

$$\Omega^m_\pm = \left(\frac{\chi_\pm + \bar\chi_\pm}{\sqrt{2}}, \quad -i\frac{\chi_\pm - \bar\chi_\pm}{\sqrt{2}}, \quad i\frac{\tilde\chi_\pm - \bar{\tilde\chi}_\pm}{\sqrt{2}}\right), \qquad \Omega^\gamma_\pm = \frac{\tilde\chi_\pm + \bar{\tilde\chi}_\pm}{\sqrt{2}\,g^2}\,. \tag{3.15}$$

(In the H-monopole case, the $\Omega^m_\pm$ are identical because the coordinates r^m are not affected by T-duality, but $\Omega^\theta_\pm = \mp g^2\Omega^\gamma_\pm = \mp(\tilde\chi_\pm + \bar{\tilde\chi}_\pm)/\sqrt{2}$.) The vector combination $(\chi\chi)^m_\pm$ can be expressed in terms of these real fermion fields:

$$(\chi\chi)^m_\pm = i\Big(\Omega^2_\pm\Omega^3_\pm + g^2\Omega^1_\pm\Omega^\gamma_\pm\,, \quad \Omega^3_\pm\Omega^1_\pm + g^2\Omega^2_\pm\Omega^\gamma_\pm\,, \quad \Omega^1_\pm\Omega^2_\pm + g^2\Omega^3_\pm\Omega^\gamma_\pm\Big)\,. \tag{3.16}$$

The final step required before we can write the action in real superfield form is to find the appropriate fermionic partner for the gauge invariant coordinate $\kappa = \gamma + \alpha$. We have already found γ's partner $\Omega^\gamma_\pm$ above, but finding the fermionic partner for α is more subtle. It can be derived (up to an unimportant constant offset) by taking the supersymmetry variation of eq. (3.12) and solving for $\delta_\xi\alpha$, giving the result

$$\delta_\xi\alpha = \sqrt{2}\,\xi^\alpha\frac{1}{2r}\left(g^2\Omega^\gamma_\alpha - r\boldsymbol{\omega}\cdot\boldsymbol{\Omega}_\alpha\right).$$

This combines neatly with $\delta_\xi \gamma = \sqrt{2}\, \xi^\alpha \Omega^\gamma_\alpha$ to yield $\delta_\xi \kappa \equiv \sqrt{2}\, \xi^\alpha \Omega^4_\alpha$. To substitute Ω^4_α for Ω^γ_α in the expressions above, we can solve for the latter:

$$g^2 \Omega^\gamma_\alpha = H^{-1} \left(\Omega^4_\alpha + \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\Omega}_\alpha \right). \tag{3.17}$$

When the real fermions are substituted in the kinetic terms of eq. (3.13), the result is

$$\begin{aligned} iH\big(\bar{\chi}_\pm \partial_\mp \chi_\pm + \bar{\tilde{\chi}}_\pm \partial_\mp \tilde{\chi}_\pm\big) = & \frac{i}{2} H \left(\boldsymbol{\Omega}_\pm \cdot \partial_\mp \boldsymbol{\Omega}_\pm \right) + \frac{i}{2} H^{-1} \left(\Omega^4_\pm + \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\Omega}_\pm \right) \times \\ & \times \left(\partial_\mp \Omega^4_\pm + \frac{1}{2} \boldsymbol{\omega} \cdot \partial_\mp \boldsymbol{\Omega}_\pm \right) + \mathcal{O}(\Omega\Omega\partial r)\,. \end{aligned} \tag{3.18}$$

The final term results from $\partial_\mp \boldsymbol{\omega}$ and contributes to the connection terms in the nonlinear sigma model action, which we will not compute in detail.

In components, the general supersymmetric nonlinear sigma model action is [13]:

$$\begin{aligned} S = \frac{1}{2\pi} \int d^2x \Big[& -\frac{1}{2} g_{mn} \partial_\mu \phi^m \partial^\mu \phi^n - \frac{1}{2} B_{mn} \epsilon^{\mu\nu} \partial_\mu \phi^m \partial_\nu \phi^n + \\ & + \frac{i}{2} g_{mn} \Omega^m_- D_+ \Omega^n_- + \frac{i}{2} g_{mn} \Omega^m_+ D_- \Omega^n_+ + \frac{1}{4} R_{mnpq} \Omega^m_+ \Omega^n_+ \Omega^p_- \Omega^q_- \Big]\,. \end{aligned} \tag{3.19}$$

Here, $D_\pm \equiv D^\pm_0 \pm D^\pm_1$, where $D^\pm_\mu$ is the covariant derivative defined with positive or negative torsion, respectively. R_{mnpq} is the Riemann tensor defined with positive torsion. (Our conventions differ somewhat from those of [13], including the overall normalization $1/2\pi$.)

Comparing this to our component action in eq. (3.13) with the substitutions from eqs. (3.16) and (3.18), we see that the form of the kinetic terms agrees and that the metric is precisely that for the Kaluza-Klein monopole given in Eq. (2.1). The Riemann tensor components extracted from the 4-fermion terms agree with those computed from the metric as well.

4. Worldsheet instanton corrections

Worldsheet instantons in the H-monopole gauged linear sigma model have been analyzed by Tong [8]. As shown above, the gauged linear sigma model for the T-dual Kaluza-Klein monopole is very similar, so we will closely follow Tong's approach in this section. When the arguments and calculations are identical in both cases (up to differences in conventions), we will simply cite his results.

In carrying out the calculation we will use the language of classical vacua with fixed values of the moduli. Of course, strictly speaking, there is no such thing in $1+1$ dimensions as there is no symmetry breaking by the Mermin-Wagner-Coleman theorem [14, 15]. As in previous treatments, we work in the framework of the Born-Oppenheimer approximation, where fast or high-momentum modes are integrated out to give a low-energy description in terms of a quantum corrected moduli space.

4.1 The classical action for instanton sectors

In gauged linear sigma models, worldsheet instantons correspond to vortices of the gauge field. We count these by

$$k = -\frac{1}{2\pi}\int F_{12}\,.$$

These instantons include not only any holomorphic worldsheet embeddings apparent from the theories' low energy nonlinear sigma model limits but also what Witten [9] calls "point-like instantons". As neither the flat target space of the H-monopole nor the Taub-NUT target space of a single Kaluza-Klein monopole contain holomorphic two-cycles, it is these point-like instantons that are relevant here.

The first step in finding the instanton action is to identify a classical solution to represent each instanton sector $k \neq 0$. Our starting point is the bosonic component form of the gauged linear sigma model action given in eqs. (3.4)–(3.5). We must also consider the effect of $\mathcal{L}_{\text{top.}} = \epsilon^{\mu\nu}\partial_\mu(\theta A_\nu) = -\theta F_{01} + \epsilon^{\mu\nu}\partial_\mu\theta A_\nu$. The first term is topologically significant, and as discussed below eq. (3.14) the impact of the second term is unclear because the action does not treat θ as a dynamical field. In the H-monopole case, θ is constant in the $g \to 0$ limit taken below, and we will make that assumption here.

To begin the calculation, we choose a specific classical vacuum for our instanton solution to approach at large distance. All vacua must satisfy $\phi = \sigma = 0$, and we can use the SU(2) R-symmetry to set $\tilde{q} = 0$ without loss of generality. The vacuum conditions in this case are $r^1 = r^2 = 0$ and $|q|^2 = r^3 \equiv \zeta$, where we define ζ as a constant parameterizing the vacuum.

Of course, the Mermin-Wagner-Coleman theorem implies that $\zeta = r^3$ cannot be a true modulus, and indeed there are no finite action solutions of the equations of motion satisfying these vortex boundary conditions. This difficulty can be overcome by an analogue of the "constrained instantons" procedure [16]: we perform our calculations in a limit of the parameters of the theory in which appropriate BPS solutions exist, and then rely on supersymmetry to protect the results as we return to general parameter values. For both the H-monopole and Kaluza-Klein monopole, an appropriate limit is to take the Taub-NUT radius $g \to 0$. This procedure is justified by the final result of the calculation, in which instanton corrections are finite even in the strict $g \to 0$ limit.

The next step is to identify the significant bosonic variations about this chosen vacuum. Only variations that could affect the gauge field are relevant; others will merely increase the total Euclidean action. We can exclude variations in ϕ and σ from the start, and variations in $\tilde{q}$, r^1, and r^2 are related by R-symmetry to variations of q and r^3 and will not reduce the action. After Wick rotating to Euclidean space (with $x^2 \equiv ix^0$) the remaining action is

$$S_{\mathrm{E}} = \frac{i}{2\pi}\int d^2x\left[\frac{F_{12}^2}{2e^2} + \frac{1}{2g^2}\left(\partial_\mu r^3\right)^2 + \frac{g^2}{2}(\partial_\mu\gamma + A_\mu)^2 + |\mathcal{D}_\mu q|^2 + \frac{e^2}{2}\left(|q|^2 - r^3\right)^2 + i\theta F_{12}\right].$$

When $g \to 0$, the γ kinetic term drops out of the action entirely. On the other hand, variations of r^3 away from ζ are frozen out when $g \to 0$, even when $|q|^2 \neq \zeta$.

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Thus, the relevant action for the instanton calculation is

$$S=\frac{i}{2\pi}\int d^2x\left[\frac{1}{2e^2}F_{12}^2+|\mathcal{D}_\mu q|^2+\frac{e^2}{2}\left(|q|^2-\zeta\right)^2+i\theta F_{12}\right].$$

This is precisely the same action as in the H-monopole case: the abelian Higgs model action at critical coupling plus a θ term. Completing the square gives

$$S=\frac{i}{2\pi}\int d^2x\left[\frac{1}{2e^2}\left(F_{12}\mp e^2(|q|^2-\zeta)\right)^2+|\mathcal{D}_1 q\pm i\mathcal{D}_2 q|^2+\left(\mp\zeta+i\theta\right)F_{12}\right].$$

The first two terms are strictly non-negative, so the minimal action occurs when they are zero. This provides a set of first order Bogomol'nyi equations for the vortex solution:

$$F_{12}=\pm e^2(|q|^2-\zeta)\qquad\text{and}\qquad \mathcal{D}_{\bar{z}}q=0\quad(\text{or lower sign: }\mathcal{D}_z q=0)\,.$$

When integrated, the third term is proportional to the instanton number k. Choosing the $\pm$ sign to give the tightest lower bound on the real part of the action (the top sign is preferred when $k>0$), we find that the action when the Bogomol'nyi equations are satisfied is

$$S_k=|k|\zeta-ik\theta\,, \tag{4.1}$$

where we have defined $S_k=-iS$ in the given instanton sector (so the path integral factor e^{iS} becomes e^{-S_k}).

4.2 The instanton sum and measure

The sum over instanton configurations has two parts: a discrete sum over sectors k (each represented by a solution $\{A_\mu^{(k)},q^{(k)}\}$ of the Bogomol'nyi equations) and an integral over zero modes. (Tong argues that the contributions of bosonic and fermionic non-zero modes cancel in the present case.) To evaluate this integral, we must find the proper measure by identifying the bosonic and fermionic zero modes of the solution. We must also identify any corrections to the instanton action that depend on the zero modes and find the long distance behavior of the zero mode solutions themselves. As the action here matches the H-monopole case, this section simply summarizes the results of [8] except as noted.

The first step is to identify the proper measure for the bosonic zero mode integral. We begin from the linearized Bogomol'nyi equations, which together with a gauge fixing condition can be written as a bosonic Dirac equation. For $k>0$,

$$\Delta\begin{pmatrix}\delta A_{\bar{z}}\\ \delta q\end{pmatrix}=0\,,\qquad\text{where}\qquad \Delta\equiv\begin{pmatrix}\frac{2i}{e^2}\partial & q^\dagger\\ -q & i\overline{\mathcal{D}}\end{pmatrix}.$$

Erick Weinberg [17, 18] used index theory to show that these equations have $2|k|$ normalizable, linearly independent zero mode solutions. These form the multi-vortex moduli space $\mathcal{M}_k$, which decomposes as $\mathcal{M}_k=\mathbb{R}^2\times\tilde{\mathcal{M}}_k$. The coordinates X^μ on $\mathbb{R}^2$ are Goldstone modes encoding the center of mass of the vortices; for $k>0$ and $\mu=1,2$ the corresponding linearized fields after gauge fixing are

$$\{\delta_\mu A_\nu=F_{\mu\nu}\,,\qquad \delta_\mu q=\mathcal{D}_\mu q\}\,. \tag{4.2}$$

The coordinates Y^p on $\tilde{\mathcal{M}}_k$ ($p=1,\ldots,2(k-1)$) encode the relative vortex positions.

The metric on $\mathcal{M}_k$ is defined by the overlap of the zero modes [19]; the proper overlap integral emerges from a standard gauge-fixed zero mode calculation [20]. This can be computed explicitly for the Goldstone modes above, for which we find $g_{\mu\nu} = \zeta|k|\delta_{\mu\nu}$. The metric $\tilde{g}_{pq}$ on $\tilde{\mathcal{M}}_k$ remains unknown for any $|k| > 1$. The bosonic zero mode integral is

$$\int d\mu_B = \int d^2X \prod_{p=1}^{2(|k|-1)} dY^p \, \frac{\sqrt{\det g}}{(2\pi)^{|k|}} = \frac{\zeta|k|}{2\pi} \int d^2X \prod_{p=1}^{2(|k|-1)} dY^p \, \frac{\sqrt{\det \tilde{g}}}{(2\pi)^{|k|-1}} \,. \tag{4.3}$$

Next, we require the corresponding measure for the fermionic zero modes. There are $4k$ of these, related to the $2k$ bosonic zero modes by the unbroken supersymmetries. The superpartners of the X^μ are Goldstino modes from the broken supersymmetries. Two of these result from the breaking of our explicit $\mathcal{N} = (2,2)$ supersymmetry and are parameterized by the Grassmann variables α_1 and α_2, while the other two are related to these by R-symmetry and are parameterized by $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. Explicitly, these are pairs of fermion fields which are zero modes of the operator Δ or its complex conjugate Δ^* (neither $\Delta^\dagger$ nor Δ^T have any zero modes). For $k > 0$:

$$\begin{array}{cccc} \frac{i}{\sqrt{2}}\bar{\lambda}_+ = \frac{i}{2}\alpha_1 F_{12}\,, & -\frac{i}{\sqrt{2}}\lambda_- = -\frac{i}{2}\alpha_2 F_{12}\,, & -\frac{1}{\sqrt{2}}\tilde{\lambda}_+ = \frac{i}{2}\tilde{\alpha}_1 F_{12}\,, & -\frac{1}{\sqrt{2}}\bar{\tilde{\lambda}}_- = -\frac{i}{2}\tilde{\alpha}_2 F_{12} \\ \psi_- = \alpha_1 \mathcal{D}q\,, & \bar{\psi}_+ = \alpha_2 \overline{\mathcal{D}} q^\dagger\,, & \bar{\tilde{\psi}}_- = \tilde{\alpha}_1 \mathcal{D}q\,, & \tilde{\psi}_+ = \tilde{\alpha}_2 \overline{\mathcal{D}} q^\dagger \quad . \end{array} \tag{4.4}$$

These results differ somewhat from those given in [8] for this action, and we show that our ψ and $\tilde{\psi}$ zero modes lead to a simpler expression for the four-fermion correlation function when $|k| > 1$. As in the bosonic case, no explicit form is known for the fermion zero mode partners of the relative coordinates Y^p; we parameterize them by β^p and $\tilde{\beta}^p$.

The overlap integrals that define the fermionic moduli space metric g arise from a zero mode calculation analogous to the bosonic case. The result is the same apart from a shift in normalization, which can be thought of as a change of zero mode basis in eq. (4.2) from $\mu = 1, 2$ to $\mu = z, \bar{z}$ to match eq. (4.4). The measure for the fermion zero mode integral is

$$\int d\mu_F = \int d^2\alpha \, d^2\tilde{\alpha} \prod_{p=1}^{2(|k|-1)} d\beta^p \, d\tilde{\beta}^p \frac{1}{\det g} = \left(\frac{2}{\zeta|k|}\right)^2 \int d^2\alpha \, d^2\tilde{\alpha} \prod_{p=1}^{2(|k|-1)} d\beta^p \, d\tilde{\beta}^p \frac{1}{\det \tilde{g}} \,. \tag{4.5}$$

While the zero modes found above each solve the linearized equations of motion, they may interfere with each other when integrated up to solutions of the full system. This results in a four-fermion contribution to the action by relative fermion zero modes [21, 22]:

$$S_{\text{4-fermi}} = \frac{1}{4} \tilde{R}_{pqrs} \beta^p \beta^q \tilde{\beta}^r \tilde{\beta}^s \,. \tag{4.6}$$

Here, $\tilde{R}$ is the Riemann tensor on the relative vortex moduli space $\tilde{\mathcal{M}}_k$.

Finally, we need to know the explicit long-distance limit of the Goldstino mode solutions. The long-distance limit of the bosonic field q was found in [23]. Using polar coordinates on the Euclidean worldsheet, $z = \rho e^{i\vartheta}$, the solution for a k-vortex solution centered at the origin as $\rho \to \infty$ is

$$|q|^2 \to \zeta \left(1 - l_k(Y^p, \vartheta)\sqrt{\frac{2\pi L}{\rho}}\, e^{-\rho/L}\right),$$

where the characteristic vortex length scale is $L = (2e^2\zeta)^{-1/2}$. The functions $l_k(Y^p, \vartheta)$ are unknown except for the numerical constant $l_1 = 8^{1/4}$ [8]. The phase of q in the vortex solution is important: in our conventions, $q = |q|e^{ik\vartheta}$. We also need to know the corresponding $\rho \to \infty$ limit of the gauge field, $A_{\bar{z}} \to \frac{i}{2}e^{i\vartheta}(-k/\rho + l_k(Y^p, \vartheta)\sqrt{\pi/2L\rho}\; e^{-\rho/L})$, which together with this q satisfies $\overline{\mathcal{D}}q = 0$ to the given order in ρ.

From these results we can find the long-distance behavior of the Goldstino mode ψ_-; the others will have either the same profile or its conjugate. For $k > 0$,

$$\psi_- = \alpha_1 \mathcal{D} q \to \alpha_1 \sqrt{\zeta}\; l_k(Y^p, \vartheta)\; e^{i(k-1)\vartheta} \sqrt{\frac{\pi}{2L\rho}}\; e^{-\rho/L}\,. \tag{4.7}$$

The final square root and exponential will be denoted $S_F(X)$ below, as they give the asymptotic behavior of the diagonal component of a Dirac fermion propagator with mass $1/L$.

4.3 Instanton corrections to the geometry

We can now assemble these results to compute the instanton contribution to the ψ^4 correlation function. This will correspond to a modified four-fermion vertex for ψ in the low energy effective action. As these modifications must still be of the geometric form in eq. (3.19), they can be interpreted as correcting the Riemann curvature tensor for the monopole geometry. The four insertions must be able to absorb all four fermionic zero mode integrals, so the only non-vanishing set of insertions in the k-instanton sector when $k > 0$ is

$$\begin{aligned} G_4^{(k)}(x_1, x_2, x_3, x_4) &= \left\langle \bar{\psi}_+(x_1)\psi_-(x_2)\tilde{\psi}_+(x_3)\bar{\tilde{\psi}}_-(x_4)\right\rangle\Big|_{k\text{-instanton}} \\ &= \int d\mu_B d\mu_F \left[\bar{\psi}_+(x_1)\psi_-(x_2)\tilde{\psi}_+(x_3)\bar{\tilde{\psi}}_-(x_4)e^{-S_k - S_{4-\mathrm{fermi}}}\right]. \end{aligned}$$

If $k < 0$, the conjugate holds. The components of this expression can be found in eqs. (4.1), (4.3), (4.5), (4.6), and (4.7), and together they give

$$\begin{aligned} G_4^{(k)}(x_1, x_2, x_3, x_4) &= \frac{1}{(2\pi)^{d/2}} \frac{2\zeta}{\pi|k|} e^{-|k|\zeta + ik\theta} \times \\ &\times \int d^2X \prod_{p=1}^{d} \left(dY^p d\beta^p d\tilde{\beta}^p\right) \frac{l_k^4(Y^p, \vartheta)}{\sqrt{\det \tilde{g}}} e^{-\frac{1}{4}\tilde{R}_{pqrs}\beta^p\beta^q\tilde{\beta}^r\tilde{\beta}^s} \prod_{i=1}^{4} S_F(X - x_i)\,. \end{aligned}$$

Here, $d = 2(|k| - 1)$ is the dimension of the relative moduli space. The worldsheet position appears here only in the propagator terms (which we trust only when the $|X - x_i|$ are large) and in the ϑ dependence of $l_k^4(Y^p, \vartheta)$, which characterize the vortex solution falloff at large distance. We expand any such dependence on ϑ as a Taylor series and proceed using only the term without higher derivative corrections: the ϑ-averaged value of $l_k^4(Y^p, \vartheta)$. We can then separate out all terms involving the relative moduli space into a function $\nu(\tilde{\mathcal{M}}_k)$:

$$G_4^{(k)}(x_1,x_2,x_3,x_4) = \frac{2\zeta}{\pi|k|}e^{-|k|\zeta+ik\theta}\nu(\tilde{\mathcal{M}}_k)\int d^2X\prod_{i=1}^{4}S_F(X-x_i)\,.$$

Explicitly, the function $\nu(\tilde{\mathcal{M}}_k)$ is

$$\begin{aligned}\nu(\tilde{\mathcal{M}}_k) &= \frac{1}{(2\pi)^{d/2}}\int\left[\prod_{p=1}^{d}\left(dY^p d\beta^p d\tilde{\beta}^p\right)\frac{e^{-\frac{1}{4}\tilde{R}_{pqrs}\beta^p\beta^q\tilde{\beta}^r\tilde{\beta}^s}}{\sqrt{\det\tilde{g}}}\frac{1}{2\pi}\int d\vartheta\, l_k^4(Y^p,\vartheta)\right]\\ &= \frac{1}{(-8\pi)^{d/2}(d/2)!}\int\prod_{p=1}^{d}dY^p\sqrt{\det\tilde{g}}\,\epsilon^{p_1p_2\ldots p_d}\epsilon^{q_1q_2\ldots q_d}\tilde{R}_{p_1p_2q_1q_2}\cdots\tilde{R}_{p_{d-1}p_dq_{d-1}q_d}\times\\ &\quad\times\frac{1}{2\pi}\int d\vartheta\, l_k^4(Y^p,\vartheta)\,.\end{aligned}$$

Note that $\epsilon^{\cdots}$ is the usual contravariant volume element whose non-zero components have magnitude $1/\sqrt{\det\tilde{g}}$. For $|k|=1$ no calculation is necessary: $\nu = l_1^4$. For higher $|k|$, this expression differs from the same result in [8]: the phases of the four fermion zero modes cancel out, so there is no exponential of $i\vartheta$ weighting the exponential falloff function. Thus, this is simply the integral of the Euler form over the k-vortex moduli space, weighted by the average exponential falloff.

The final modification to the four-fermion term in the low energy effective action is found from the sum over instanton sectors, $-\sum_k G_4^{(k)}$ (the minus sign appears after Wick rotating back to a lorentzian worldsheet). Restoring r in place of ζ, we obtain (up to a possible unimportant numerical factor)

$$\begin{aligned}\delta\mathcal{L}_{\text{eff}} &= -\sum_{k=1}^{\infty}\frac{2r}{\pi|k|}\nu(\tilde{\mathcal{M}}_k)e^{-kr}\left[e^{ik\theta}\bar{\psi}_+\psi_-\tilde{\psi}_+\bar{\tilde{\psi}}_- + e^{-ik\theta}\bar{\psi}_-\psi_+\tilde{\psi}_-\bar{\tilde{\psi}}_+\right]\\ &= -\sum_{k=1}^{\infty}\frac{1}{2\pi|k|r}\nu(\tilde{\mathcal{M}}_k)e^{-kr}\left[e^{ik\theta}\tilde{\chi}_+\bar{\tilde{\chi}}_-\chi_+\bar{\chi}_- + e^{-ik\theta}\tilde{\chi}_-\bar{\tilde{\chi}}_+\chi_-\bar{\chi}_+\right]\\ &= -\sum_{k=1}^{\infty}\frac{1}{8\pi|k|r}\nu(\tilde{\mathcal{M}}_k)e^{-kr}\Bigg[\ e^{ik\theta}\,(\Omega_+^1+i\Omega_+^2)\left(\Omega_+^3+iH^{-1}\left(\Omega_+^4+\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\Omega}_+\right)\right)\times\\ &\qquad\times(\Omega_-^1-i\Omega_-^2)\left(\Omega_-^3-iH^{-1}\left(\Omega_-^4+\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\Omega}_-\right)\right)+\\ &\qquad+e^{-ik\theta}(\Omega_+^1-i\Omega_+^2)\left(\Omega_+^3-iH^{-1}\left(\Omega_+^4+\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\Omega}_+\right)\right)\times\\ &\qquad\times(\Omega_-^1+i\Omega_-^2)\left(\Omega_-^3+iH^{-1}\left(\Omega_-^4+\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\Omega}_-\right)\right)\Bigg]\\ &\xrightarrow{g\to 0} -\sum_{k=1}^{\infty}\frac{1}{8\pi|k|r}\nu(\tilde{\mathcal{M}}_k)e^{-kr}\Big[e^{ik\theta}(\Omega_+^1+i\Omega_+^2)\Omega_+^3(\Omega_-^1-i\Omega_-^2)\Omega_-^3+\\ &\qquad+e^{-ik\theta}(\Omega_+^1-i\Omega_+^2)\Omega_+^3(\Omega_-^1+i\Omega_-^2)\Omega_-^3\Big]\,.\end{aligned}$$

As this is part of the low energy action, we have used eq. (3.9) and our vacuum choice $\tilde{q}=0$ in deriving the second line and eqs. (3.15) and (3.17) to find the third. In taking the final

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limit, we have dropped terms involving $H^{-1} \propto g^2$ as our calculation may have neglected terms of this same order. By comparison with eq. (3.19) (and after accounting for the symmetries of the Riemann tensor) we can see that the net coefficient of $\Omega_+^m \Omega_+^n \Omega_-^p \Omega_-^q$ here is $\delta R_{mnpq}/(2\pi)$.

This result can be extended to $r^1, r^2 \neq 0$ by R-symmetry (corresponding to rotations in target space), but by construction it will hold only in a particular large distance limit. We have kept only the leading term at large r for each instanton sector, but we sum over higher $|k|$ modes despite their exponential suppression. Treating the small quantities $1/r$ and e^{-r} independently in this way is physically reasonable because the exponential terms have a distinct origin as higher instanton sectors.

For the terms which survive the final $g \to 0$ limit, this result for $\delta\mathcal{L}$ is identical to the H-monopole case: as noted previously, the low energy component constraints in eq. (3.9) and the real scalar superpartners $\Omega_\pm^{1,2,3}$ are the same for both monopole solutions. Thus, the leading corrections to the components of the Kaluza-Klein monopole Riemann tensor with indices 1, 2, and 3 must be the same as those for the localized H-monopole.

To leading order in $1/r$, it can be verified that the Riemann tensor with torsion is

$$R_{mnpq} = -\frac{1}{2}(\partial_m\partial_p g_{nq} + \partial_n\partial_q g_{mp} - \partial_m\partial_q g_{np} - \partial_n\partial_p g_{mq}) - \frac{1}{2}(\partial_m T_{npq} - \partial_n T_{mpq})\,.$$

This assumes that both $\Gamma^m{}_{pq}$ and $T^m{}_{pq}$ fall off at least as $1/r$, which does hold in our case. Applying these formulas to the localized H-monopole geometry given in section 2, we can (for instance) find the curvature corrections to leading order in $1/r$ evaluated at $r^1 = r^2 = 0$ for comparison to the instanton result above:

$$\begin{aligned}\delta R_{1313} = \quad \delta R_{2323} &= -\frac{1}{4r}\sum_{k=1}^{\infty} k^2 e^{-kr}(e^{ik\theta} + e^{-ik\theta}) = -\frac{1}{2r}\sum_{k=1}^{\infty} k^2 e^{-kr}\cos(k\theta)\\ \delta R_{1323} = -\delta R_{2313} &= \quad \frac{i}{4r}\sum_{k=1}^{\infty} k^2 e^{-kr}(e^{ik\theta} - e^{-ik\theta}) = -\frac{1}{2r}\sum_{k=1}^{\infty} k^2 e^{-kr}\sin(k\theta)\,.\end{aligned}$$

Naturally, we need not limit ourselves to $r^1 = r^2 = 0$ in general. The physical meaning of these corrections is different in the two cases: in the Kaluza-Klein monopole, θ is now the dyonic coordinate rather than a part of the geometry. In interpreting these results, it is useful to recall that $R_{pqmn} = R_{mnpq}|_{T\to -T}$ whenever the torsion is a closed form (which always holds in string theory, where $\mathsf{T} = -d\mathsf{B}$). This allows us to recognize that the terms in the first line must come entirely from corrections to the metric while the terms in the second line must come from an instanton-induced torsion.

While many geometries would have this same limit at large distance, the connection to the localized H-monopole's Riemann tensor corrections suggests that the metric corrections are the same:

$$\delta g_{11} = \delta g_{22} = \delta g_{33} = \frac{1}{2r}\sum_{k=1}^{\infty} e^{-kr}\left(e^{ik\theta} + e^{-ik\theta}\right). \tag{4.8}$$

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This corresponds to a correction to the harmonic function H in those metric components to the form in Eq. (2.3):

$$H = \frac{1}{g^2} + \frac{1}{2r} \sum_{k=-\infty}^{\infty} e^{-|k|r+ik\theta} = \frac{1}{g^2} + \frac{1}{2r} \frac{\sinh r}{\cosh r - \cos\theta} .$$

It seems likely that in the full corrected solution (beyond our $g \to 0$ limit), the harmonic function is modified in this way in all components of the Kaluza-Klein monopole metric. Whether there are additional corrections is unclear.

The instanton corrections also generate a torsion, in contrast to the usual Kaluza-Klein monopole solution. The only component that is non-zero to first order in $1/r$ in the $g \to 0$ limit is

$$T_{123} = -H_{123} = -\frac{1}{r} \sum_{k=1}^{\infty} k e^{-kr} \sin(k\theta) . \tag{4.9}$$

This may be written in terms of the localized harmonic function $H(r, \theta)$ as $H_{123} = \partial_\theta H$.

5. Interpretation and conclusions

5.1 Winding space localization of the Kaluza-Klein monopole

To understand these corrections, we must return to the conjecture of [7] that the proper Kaluza-Klein monopole in string theory should have some sort of "throat" behavior, just as the NS5-brane does. (Strictly speaking, a throat is only present for higher monopole charge, but there are hints of it even in the poorly understood unit charge case.) In particular, that paper suggested that just as the H-monopole throat can be probed by strings with momentum along θ, the Kaluza-Klein monopole throat could be probed by strings winding around κ. Meanwhile, the geometrical isometry along κ remains unbroken.

As part of that work, [7] studied the behavior of winding strings in the Kaluza-Klein dyon geometry. That analysis showed that although strings can unwind from the κ circle in various ways, a generalization of the winding charge remains conserved. Each change in string winding number is offset by a finite shift in the "velocity" $\dot{\beta}(t)$, where $\beta(t)$ is the dyonic coordinate introduced in eq. (2.2). Intuitively, β is the coordinate on "winding space", and string winding charge is equivalent to momentum in β.

As seen in eq. (3.14), after T-duality from the H-monopole the role of this dyonic coordinate is played by θ: "momentum space" has become "winding space". The corrections found above give strong evidence that the conjectured localization and throat behavior do appear. The modified harmonic function $H(r, \theta)$ has the same form that described a throat in the H-monopole, but it now appears only for a special value of the winding space coordinate θ rather than at a special point around the geometrical circle. And as expected from duality, the resulting torsion provides a mechanism for this structure to couple to winding strings.

Our interpretation of the corrections to the Kaluza-Klein monopole solution differs somewhat from that of [7]. That paper viewed this winding space localization as a coherent state of classical string winding modes, in analogy with an interpretation of the localized H-monopole as a coherent state of string momentum modes. Intuitively, this picture is exactly right: the localized monopole solutions can be expanded in Fourier modes that carry the correct conserved charges. However, the classical solutions for strings with momentum and winding are known, and superpositions of those solutions with the weights predicted by [7] do not give the proper correction terms on either side of T-duality.

We expect this monopole to leave some supersymmetry unbroken just as the NS5-brane does, but at first this seems impossible. One of the conditions for unbroken supersymmetry is that the dilatino variation vanish:

$$\left(\gamma^m \partial_m \Phi - \frac{1}{6}\gamma^{mnp} H_{mnp}\right) \xi \stackrel{?}{=} 0\,.$$

For non-trivial solutions ξ to exist, the γ matrices must factor out to leave a projection operator $1 \pm \gamma^5$. This is possible only if the coefficients of γ^m and $\gamma^m\gamma^5$ are equal in magnitude for each m. In particular, for $m = 4$ this condition requires $|\partial_4 \Phi| = |H_{123}|$ (with tangent space indices). As we have found that $H_{123} \neq 0$, this holds only if $\partial_4 \Phi \neq 0$, but we have seen no physics that would break the κ isometry. (Changing from curved to tangent space indices does not solve the problem.)

The resolution to this puzzle is that the usual supergravity approximation is only expected to hold when the radius g of the κ circle is large and momentum states are light. Because we have performed the instanton calculation in the limit of small g, the proper light degrees of freedom are instead the winding states and a different low-energy theory must apply. T-duality suggests that it should formally agree with the supergravity description of the H-monopole at large radius, involving the dyonic coordinate θ rather than the geometrical coordinate κ. In particular, it seems likely that the relevant part of the equation for unbroken supersymmetry in this case will be

$$\left(\gamma^\theta \partial_\theta \Phi - \gamma^{123} H_{123} + \cdots\right) \xi = 0\,. \tag{5.1}$$

Here, γ^θ denotes the matrix γ^4 from the H-monopole, and in the $g \to 0$ limit, γ^{123} agrees with that case as well. As noted below eq. (4.9), $H_{123} = \partial_\theta H$ (with curved space indices), so if eq. (5.1) is valid the dilaton should be $e^\Phi = H(r, \theta)$ just as for the H-monopole. This would lead to a throat behavior at a particular value of θ, which we expect to persist even for finite g.

5.2 Conclusions and open questions

While the Kaluza-Klein monopole geometry is well known, its familiar form does not correspond to the full solution in string theory. The usual form is "smeared" in winding space, and worldsheet instanton effects lead to its localization there. The corrections involved are very similar to those that localize the smeared H-monopole, but they explicitly depend on the Kaluza-Klein monopole's dyonic coordinate rather than the geometrical coordinate on the circle.

This work leaves a number of interesting questions unanswered. Perhaps the most basic of these is the exact form of the corrected geometry itself: our calculation was carried out in the strict $g \to 0$ limit and only to leading order in $1/r$. A better understanding supergravity when string winding states are light could be a helpful step in that direction, and would have importance in its own right.

Another natural extension of this work is to look for similar corrections to other objects in the duality web. Kaluza-Klein monopoles also appear in M-theory, and this work would seem to suggest that those solutions receive similar corrections from membrane instantons. It could also be instructive to study the case of higher monopole charge: true throat behavior does not emerge in the H-monopole until the charge is greater than one, and the same is presumably true for Kaluza-Klein monopoles as well.

Finally, it remains clear that the winding space coordinate θ appears in the action in a fundamentally different way than the geometrical coordinate κ. While differences are certainly expected, it is odd to find that the action does not appear to specify the dynamics of θ at all. It would be valuable to develop a more symmetric description of geometrical coordinates and their duals. Such a formalism could be important in finding an appropriately generalized supergravity theory as well.

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References

[1] R. d. Sorkin, *Kaluza-Klein monopole*, *Phys. Rev. Lett.* **51** (1983) 87.

[2] D.J. Gross and M.J. Perry, *Magnetic monopoles in Kaluza-Klein theories*, *Nucl. Phys.* **B 226** (1983) 29.

[3] T. Banks, M. Dine, H. Dykstra and W. Fischler, *Magnetic monopole solutions of string theory*, *Phys. Lett.* **B 212** (1988) 45.

[4] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, *Nucl. Phys.* **B 463** (1996) 55 [hep-th/9511164].

[5] J.P. Gauntlett, J.A. Harvey and J.T. Liu, *Magnetic monopoles in string theory*, *Nucl. Phys.* **B 409** (1993) 363 [hep-th/9211056].

[6] R.R. Khuri, *A multimonopole solution in string theory*, *Phys. Lett.* **B 294** (1992) 325 [hep-th/9205051].

[7] R. Gregory, J.A. Harvey and G.W. Moore, *Unwinding strings and T-duality of Kaluza-Klein and H-monopoles*, *Adv. Theor. Math. Phys.* **1** (1997) 283 [hep-th/9708086].

[8] D. Tong, *NS5-branes, T-duality and worldsheet instantons*, *JHEP* **07** (2002) 013 [hep-th/0204186].

[9] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[10] A. Sen, *Kaluza-Klein dyons in string theory*, *Phys. Rev. Lett.* **79** (1997) 1619 [hep-th/9705212].

[11] T.H. Buscher, *Path integral derivation of quantum duality in nonlinear sigma models*, *Phys. Lett.* **B 201** (1988) 466.

[12] M. Roček and E.P. Verlinde, *Duality, quotients and currents*, *Nucl. Phys.* **B 373** (1992) 630 [hep-th/9110053].

[13] P.S. Howe and G. Sierra, *Two-dimensional supersymmetric nonlinear sigma models with torsion*, *Phys. Lett.* **B 148** (1984) 451.

[14] N.D. Mermin and H. Wagner, *Absence of ferromagnetism or antiferromagnetism in one-dimensional or two-dimensional isotropic Heisenberg models*, *Phys. Rev. Lett.* **17** (1966) 1133.

[15] S.R. Coleman, *There are no Goldstone bosons in two-dimensions*, *Commun. Math. Phys.* **31** (1973) 259.

[16] I. Affleck, *On constrained instantons*, *Nucl. Phys.* **B 191** (1981) 429.

[17] E.J. Weinberg, *Multivortex solutions of the Ginzburg-Landau equations*, *Phys. Rev.* **D 19** (1979) 3008.

[18] E.J. Weinberg, *Index calculations for the fermion-vortex system*, *Phys. Rev.* **D 24** (1981) 2669.

[19] T.M. Samols, *Vortex scattering*, *Commun. Math. Phys.* **145** (1992) 149.

[20] C.W. Bernard, *Gauge zero modes, instanton determinants and quantum- chromodynamic calculations*, *Phys. Rev.* **D 19** (1979) 3013.

[21] N. Dorey, V.V. Khoze, M.P. Mattis, D. Tong and S. Vandoren, *Instantons, three-dimensional gauge theory and the Atiyah-Hitchin manifold*, *Nucl. Phys.* **B 502** (1997) 59 [hep-th/9703228].

[22] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instantons, three-dimensional gauge theory and the Gauss-Bonnet-Chern theorem*, *Nucl. Phys.* **B 502** (1997) 94 [hep-th/9704197].

[23] H.J. de Vega and F.A. Schaposnik, *A classical vortex solution of the abelian Higgs model*, *Phys. Rev.* **D 14** (1976) 1100.

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